

DYNAMICS OF EXPONENTIAL LINEAR MAP IN FUNCTIONAL SPACE

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ABSTRACT. We consider the question of existence of a unique invariant probability distribution which satisfies some evolutionary property. The problem arises from the random graph theory but to answer it we treat it as a dynamical system in the functional space, where we look for a global attractor. We consider the following bifurcation problem: Given a probability measure μ , which corresponds to the weight distribution of a link of a random graph we form a positive linear operator Φ (convolution) on distribution functions and then we analyze a family of its exponents with a parameter λ which corresponds to connectivity of a sparse random graph. We prove that for every measure μ (*i.e.*, convolution Φ) and every $\lambda < e$ there exists a unique globally attracting fixed point of the operator, which yields the existence and uniqueness of the limit probability distribution on the random graph. This estimate was established earlier [KS81] for deterministic weight distributions (Dirac measures μ) and is known as e -cutoff phenomena, as for such distributions and $\lambda > e$ there is no fixed point attractor. We thus establish this phenomenon in a much more general sense.

1. INTRODUCTION

A dynamical system is a model of time evolution. If the asymptotic behavior of the system is independent on the initial conditions then we can say that the system forgets about its past, or that it is impossible to reconstruct the past knowing the far future. A simplest such situation arises when the system has *a fixed point which is a global attractor*, in other words that wherever we started our trajectory we land in the same spot. Our paper was motivated by studying this approach to some aspects of the theory of random graphs, which we will explain in some details after the definitions. The importance of the uniqueness of a fixed point of the dynamical system is related to the effect of decay of correlation in the underlying random graph. Specifically, if the fixed point is unique then structure in one part of the graph is asymptotically independent from such a structure in other parts of the graph. The connection between uniqueness of a fixed point and correlation was formally established by the authors in [GNS03]. The concept of correlation decay comes up frequently in statistical physics. In a particular context of Glauber dynamics on spin glasses on trees see [Mar03], [MSW03] (also [BKMP01], [BW03], and related problem of information flow on trees [Mos03]). In dynamical systems it is often connected with the existence of a unique invariant measure (or an attracting fixed point of a Perron Frobenius operator) [B00].

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The intention of this paper is to link different fields: dynamics, probability, graphs and analysis. We provide therefore detailed proofs, to make the results accessible for readers with different background.

We consider a class of nonlinear operators in the space of distributions which arises in the study of maximum weight matchings in sparse random graphs. The study of this object leads to the problem of existence and uniqueness of an invariant distribution for an iterative process and whether the iterations of any distribution converges to this invariant distribution. Our methods rely on the understanding of the dynamics of this iterative process. The phase space is however a functional space and we have a rare opportunity to study a specific non-linear system with non-trivial behavior. The nonlinear operator is a composition of an exponential map with a positive linear operator.

Definitions and main result. Let \mathcal{F} be a family of functions on the segment $I \subset \mathbb{R}$ with values in $[0, 1]$, $\mathcal{F} \ni F : I \rightarrow \mathbb{R}$. For a positive linear operator (endomorphism) Ψ on \mathcal{F} we define:

$$\mathbb{T} : \mathcal{F} \rightarrow \mathcal{F} \quad \text{by} \quad \mathbb{T}(F)(x) = \exp(-\Psi(F)(x)).$$

In the case which interests us at most as it has an application to the random graphs the linear operator Ψ is a product of the parameter $\lambda > 0$ and a convolution Φ with respect to a given probability measure μ . We restrict the domain of Φ to nondecreasing functions of an interval, which we fix here to be $[0, 1]$. Specifically: $\mathcal{D} \ni F(z) : [0, 1] \rightarrow [0, 1]$ and F is not decreasing. Given a (Borel) probability measure μ on $[0, 1]$ we define a linear operator on μ integrable functions (Lebesgue integral) by:

$$\Phi_\mu(F) = \int_x^1 F(z-x) d\mu(z) = \int_0^{1-x} F(z) d\mu(z+x),$$

and with it, for any $\lambda > 0$, an exponential map by:

$$F \mapsto \mathbb{T}_\lambda(F)(x) = \mathbb{T}F(x) = \exp(-\lambda\Phi_\mu(F)(x)).$$

The main result of this paper is an extension of the *e-cutoff phenomenon*, which was established earlier by Karp and Sipser [KS81] only for deterministic distributions. We state this result in three versions:

- Theorem 4, which carries the burden of the proof. For any positive linear Ψ : If $\|\Psi\| < e$, then \mathbb{T}^2 is a contraction.
- Theorem 5 is convolution specific. For any probability measure μ , if $\lambda < e$, then \mathbb{T} has a fixed point which is a global attractor.
- Theorem 2 restates the result in the probabilistic setting.

Applications to maximum weight matching in sparse random graphs.

Before we prove our main results we describe in more details and in the probabilistic setting the connection between the fixed point properties of the dynamical systems considered in Sections 2 and 3 and the theory of random graphs.

The following is a standard model of a sparse random graph on n nodes with average degree (connectivity) λ [Bol85], [JLR00]. Often this model is also called Erdos-Renyi graph. Given a collection of n nodes $1, 2, \dots, n$, an edge (link) (i, j) , $1 \leq i \neq j \leq n$ is selected to belong to the graph with probability λ/n , independently for

all $n(n-1)/2$ pairs i, j . The collection of selected edges is denoted by E . The selected edges are equipped with randomly generated non-negative weights $W_{i,j}$, distributed according to a common distribution function $\mathbb{P}(W_{i,j} \leq x) \equiv \mu(x), x \geq 0$. A matching is any collection of edges in E which do not share a node. That is $M \subset E$ is a matching if for every $(i_1, j_1), (i_2, j_2) \in M$ the nodes i_1, i_2, j_1, j_2 are distinct. The weight of a matching M is the sum $\sum_{(i,j) \in M} W_{i,j}$. We let $M_\mu(n, \lambda) = \max_M \sum_{i,j} W_{i,j}$ denote the maximum weight of a matching. Note that $M_\mu(n, \lambda)$ is a random variable which only depends on n, λ and the distribution function μ . The main question of interest is establishing the existence and computing the limit

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}[M_\mu(n, \lambda)]}{n},$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator. This problem was solved for the case of the deterministic weights ($W_{i,j} = 1$ with probability one (w.p.1)) by Karp and Sipser [KS81] using a simple combinatorial argument, that we reproduce for completeness in the third part of Theorem 7. The threshold $\lambda = e$ also corresponds to some phase transition property in the underlying random graph. This phase transition was thereafter called e -cutoff phenomena.

The Karp-Sipser method however does not apply to the case of non-deterministic weights and the authors [GNS03] solved the problem of computing limit (1) using the completely different Local Weak Convergence (LWC) method, developed earlier by Aldous [Ald92], [Ald01], Aldous and Steele [AS03]. The existence of the limit (1) was first established by the first author [Gam04] using a non-constructive version of LWC method and only in [GNS03] we were able to compute the limit at least for some non-deterministic distribution. The method is heavily based on solving for fixed point solutions of certain distributional equations. We give here only a quick description of the main result in [GNS03] regarding matching and refer the reader to the paper for further details.

Let $F = F(x)$ be a distribution function corresponding to some non-negative random variable, and let K be a random variable distributed according to the Poisson distribution with parameter λ , denoted $\text{Pois}(\lambda)$. That is $\mathbb{P}(K = k) = (\lambda^k/k!)e^{-\lambda}$. Consider a random variable $X = \max_{i \leq K} (W_i - X_i)$, where X_1, \dots, X_K are distributed according to F , independently and W_1, \dots, W_K are distributed according to $\mu = \mu(x)$ independently. When $K = 0$, X is assumed 0 by convention. Let \tilde{F} denote the distribution function of X . This defines an operator $F \mapsto \tilde{F}$ on the space of distribution functions, indexed by λ and the distribution function μ . We claim that this operator is in fact $F \mapsto \mathbb{T}_\lambda(F)$ defined in Section 3. Indeed:

$$\begin{aligned} \mathbb{P}(X \leq x) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} (\mathbb{P}(W_1 - X_1 \leq x))^k \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \left(\mu(x) + \int_x^{\infty} (1 - F(z - x)) d\mu(z) \right)^k \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\mu(\infty) - \int_x^{\infty} F(z - x) d\mu(z) \right)^k \\ &= \exp \left(-\lambda \int_x^{\infty} F(z - x) d\mu(z) \right), \end{aligned}$$

where in the last equality we use the fact that μ is the distribution function, and therefore $\mu(\infty) = 1$. We see that the distribution \tilde{F} of X is indeed given by $\mathbb{T}_\lambda(F)$. The following theorem was established in [GNS03] (Theorem 2, Equation (9)).

Theorem 1. *Given an atom-free distribution function μ , suppose the operator \mathbb{T}_λ^2 has the unique fixed point solution $F^* = \mathbb{T}_\lambda^2(F^*)$. Then the limit (1) is equal to:*

$$(2) \quad \frac{1}{2} \mathbb{E} \left[\sum_{i \leq K} W_i \chi \{ W_i - X_i = \max_{j \leq K} (W_j - X_j) > 0 \} \right],$$

where K is distributed as $\text{Pois}(\lambda)$, W_1, \dots, W_K are distributed according to μ , and X_1, \dots, X_K are distributed according to F^* . (χA is the indicator function of the set A).

The expression above can easily be transformed into an expression involving integrals and distribution functions. Thus the theorem states that whenever the fixed point F^* of \mathbb{T}_λ^2 is unique, the maximum weight matching can be computed by computing the expectation above with respect to measures μ , F^* and $\text{Pois}(\lambda)$. The theorem then justifies the search for measures μ and parameters λ for which the corresponding operator \mathbb{T}_λ^2 has the unique fixed point. Later on Theorem 5 shows that for every distribution μ and every $\lambda < e$ the operator \mathbb{T}_λ does have a unique fixed point which is a global attractor and, as a result, \mathbb{T}_λ^2 has a unique fixed point. We then solve the problem of finding (1) whenever $\lambda < e$. Hence the e -cutoff Theorem 5 in a context of random graphs is:

Theorem 2 (e -cutoff probabilistic version). *Given a sparse random graph with connectivity $\lambda < e$ and given an atom-free distribution function μ , the limit (1) is equal to (2), where F^* is the unique fixed point of \mathbb{T}_λ .*

Outline of the paper.

- In next Section 2 we prove that the exponential linear map defines a dynamical system (Proposition 1) on real functions and prove its basic properties: monotonicity, continuity and differentiability in natural norms.
- We establish the existence of two specific limit functions and in Theorem 3 we prove (using only monotonicity) that the system has a fixed point global attractor if and only if they are equal. When the operator is continuous in the L^1 norm, the limit functions form a periodic cycle and Theorem 2 specifies in such a case that the existence of a fixed point global attractor is equivalent to the uniqueness of the fixed point for \mathbb{T}^2 .
- We prove that if the sup norm of the linear operator is smaller than e , the second iterate of the exponential map is a contraction and the main result (Theorem 4) follows.
- In Section 3 we deal with specific linear part, the convolutions. We restrict the phase space to a subset of non decreasing functions (in fact with range in $[0, 1]$) and prove that in this case the map defines the dynamical system on this set of distribution functions.
- We prove a criterion (Theorem 6) for the existence of the fixed point global attractor which is specific to the restricted system.
- In last Section 4 we present examples of the map for particular measures μ . In cases of Lebesgue measure (uniform distribution) and exponential distribution there are fixed points which are a global attractors for every

$\lambda > 0$. For completeness we also include the known case of Dirac measure where the fixed point is a global attractor if and only if $0 < \lambda \leq e$.

2. THE EXPONENTIAL-LINEAR DYNAMICS FOR POSITIVE LINEAR OPERATOR Ψ

In order to lighten the notation we will write $\mathbb{T}F$ and ΨF for $\mathbb{T}(F)$ and $\Psi(F)$.

Proposition 1 (\mathbb{T} defines a dynamical system on real functions). *For every $F \in \mathcal{F}$ we have $\mathbb{T}F \in \mathcal{F}$*

Proof. By positivity of Ψ , if $F \geq 0$ then $\Psi F \geq 0$. Hence $\exp(-\Psi F) \in [0, 1]$. \square

Remark 1. *The definition of \mathbb{T} can be extended by linearity of Ψ to any function $F : I \rightarrow \mathbb{R}$ such that $aF(x) + b$ lies in the domain of Ψ for some $a, b \in \mathbb{R}$, for example to bounded functions. We have for any such F :*

$$\mathbb{T}F \geq 0 \quad \text{and} \quad \mathbb{T}(\mathbb{T}F) \leq 1,$$

so our assumption on the range of $F \in \mathcal{F}$ is not very restrictive. \square

Let \mathbb{T}^n denote the n -th iteration of \mathbb{T} given by $\mathbb{T}^0(F)(x) = F(x)$ and $\mathbb{T}^{n+1}(F) = \mathbb{T}(\mathbb{T}^n(F))$.

Monotonicity properties.

Lemma 1. *The map \mathbb{T} is non increasing, the map \mathbb{T}^2 is non decreasing.* \square

Proof. If $F, G \in \mathcal{F}$, $F \leq G$ then $G - F \in \mathcal{F}$ and $0 \leq \mathbb{T}(G - F) \leq 1$. By linearity of Ψ we have $\mathbb{T}(G) = \mathbb{T}(F) \cdot \mathbb{T}(G - F) \leq \mathbb{T}(F)$. For \mathbb{T}^2 we apply the previous argument twice. \square

Define:

$$(3) \quad \mathbf{0}(x) = 0 \quad \text{and} \quad \mathbf{1}(x) = 1 \quad \text{for all } x \in I,$$

we have $\mathbb{T}(\mathbf{0}) = \mathbf{1}$. Denote:

$$\mathbf{0}^n = \mathbb{T}^n(\mathbf{0}) \quad \text{and} \quad \mathbf{1}^n = \mathbb{T}^n(\mathbf{1}),$$

we have clearly $\mathbf{0}^{n+1} = \mathbf{1}^n$, which symbolically defines $\mathbf{1}^{-1} = \mathbf{0}$.

Lemma 2. *For every $F \in \mathcal{F}$ and every $n \geq 0$ we have:*

$$\begin{aligned} \mathbf{0}^{2n} = \mathbf{1}^{2n-1} &\leq \mathbb{T}^{2n} F \leq \mathbf{1}^{2n} \\ \text{and} \quad \mathbf{1}^{2n} &\geq \mathbb{T}^{2n+1} F \geq \mathbf{1}^{2n+1} = \mathbf{0}^{2n+2}, \end{aligned}$$

and in particular:

$$\mathbf{0} = \mathbf{1}^{-1} \leq \dots \leq \mathbf{1}^{2n-1} \leq \mathbf{1}^{2n+1} \leq \dots \leq \mathbf{1}^{2n+2} \leq \mathbf{1}^{2n} \leq \dots \leq \mathbf{1}^0 = \mathbf{1}.$$

Proof. By definition $\mathbf{0} \leq F \leq \mathbf{1}$. Hence by Lemma 1, $\mathbf{1} = \mathbb{T}\mathbf{0} \geq \mathbb{T}F \geq \mathbb{T}\mathbf{1} = \mathbb{T}^2\mathbf{0}$ and the inequalities follow by induction. \square

Corollary 1. *There are point-wise, monotone limits*

$$\mathbf{L} = \lim \mathbf{0}^{2n} \leq \lim \mathbf{1}^{2n} = \mathbf{U}.$$

For every $F \in \mathcal{F}$ we have

$$\mathbf{L} \leq \liminf \mathbb{T}^n F \leq \limsup \mathbb{T}^n F \leq \mathbf{U}.$$

\square

Theorem 3 (Main criterion for uniqueness of the attractor). *The exponential linear dynamical system has a fixed point which is a global attractor if and only if the limit functions \mathbf{L} and \mathbf{U} are equal.*

Proof. If $\mathbf{L} = \mathbf{U}$ then by Corollary 1 every $\mathbb{T}^n F$ converges point-wise to a common limit. If $\mathbf{L} \neq \mathbf{U}$ then $\mathbb{T}^n \mathbf{1}$ (and any function contained between two odd or two even iterates of $\mathbf{1}$) do not converge to a limit as it has two distinct accumulation points \mathbf{L} and \mathbf{U} . \square

Remark 2. *If $\mathbf{L} \neq \mathbf{U}$ in Theorem 3 then the a global attractor (which is not a fixed point and may not be minimal) is contained in the set $\{F \in \mathcal{D} : \mathbf{L} \leq F \leq \mathbf{U}\}$ and contains both \mathbf{L} and \mathbf{U} . It seems that both inclusions are proper. It still may happen that some functions from between \mathbf{L} and \mathbf{U} converge to a fixed point.* \square

Continuity and differentiability.

From now on we assume that the linear operator is continuous in either the sup norm $\|\cdot\|_\infty$ or in the $L^1(dx)$ norm $\|\cdot\|_1$ on I . In the second case, when the segment I is infinite we assume that every $F \in \mathcal{F}$ has a bounded integral; moreover all equalities of the functions are meant in the norm sense, i.e., $F = G$ means $\|G - F\|_1 = 0$.

Lemma 3. *If Ψ is continuous (or in other words when its norm is bounded) then \mathbb{T} is continuous.*

Proof. It follows from the composition rule. \square

Remark 3. *If \mathbb{T} is continuous in some norm and the sequence $\mathbf{1}^{2n}$ converges in the same norm, then $\mathbf{1}^{2n+1}$ converges and the limits are \mathbf{U} and \mathbf{L} respectively. In such a case:*

$$\mathbb{T}\mathbf{L} = \mathbf{U} \quad \text{and} \quad \mathbb{T}\mathbf{U} = \mathbf{L}.$$

In the L^1 norm the existence of the limit is assured by the Lebesgue Convergence Theorems (either monotone or majorized).

Corollary 2 (The L^1 norm). *Suppose that $\mathbb{T}\mathbf{U} = \mathbf{L}$ and $\mathbb{T}\mathbf{L} = \mathbf{U}$. Then \mathbb{T} has a fixed point which is a global attractor if and only if \mathbb{T}^2 has a unique fixed point.*

Proof. Both \mathbf{L} and \mathbf{U} are fixed points of \mathbb{T}^2 and the result follows from Theorem 3. \square

Lemma 4. *If Ψ is continuous then the operator \mathbb{T} is differentiable with respect to F and its derivative is:*

$$D\mathbb{T}(F)(H)(x) = \mathbb{T}F(x) \cdot \Psi H(x).$$

Proof. The linear operator Ψ is continuous and hence differentiable. The formula is an application of the Chain Rule. \square

Corollary 3. *The derivative of \mathbb{T} is uniformly bounded for $F \in \mathcal{F}$ by the norm of Ψ and:*

$$\|\mathbb{T}(F + H) - \mathbb{T}(F)\| \leq \|\Psi\| \|H\|.$$

Proof. We have $|\mathbb{T}F(x) \cdot \Psi H(x)| \leq \sup |\mathbb{T}F| \cdot |\Psi H(x)| \leq |\Psi H(x)|$. The formula follows from the Mean Value Theorem. \square

Corollary 4. *If $\|\Psi\| < 1$ then \mathbb{T} has a unique fixed point which is a global attractor.*

Proof. In this case \mathbb{T} is a contraction, hence \mathbf{U} and \mathbf{L} cannot stay away and we use Theorem 3. (Note that we do not need to prove that \mathcal{F} is complete.) \square

Theorem 4. *If $\|\Psi\|_\infty < e$ then \mathbb{T} has a unique fixed point which is a global attractor.*

Proof. It is enough to show that \mathbb{T}^2 is a contraction in $\|\cdot\|_\infty$ hence then again \mathbf{U} and \mathbf{L} cannot be different. We use the Chain Rule to calculate the derivative of the second iterate:

$$D(\mathbb{T}^2)(F)(H) = \mathbb{T}^2 F \cdot \Psi(\mathbb{T}F \cdot \Psi H),$$

Since for $G > 0$ and any k we have $\Psi(G \cdot (\|k\|_\infty - k)) \geq 0$, then $|\Psi(G \cdot k)| \leq |\Psi G| \cdot \|k\|_\infty$ and therefore:

$$|D\mathbb{T}^2(F)(H)| \leq \mathbb{T}^2 F \cdot \Psi(\mathbb{T}F) \cdot \|\Psi H\|_\infty \leq \frac{\|\Psi\|_\infty}{e} \cdot \|H\|_\infty,$$

where we used the fact that $xe^{-x} \leq 1/e$ for $0 \leq x = \Psi(\mathbb{T}F)$. Now from the Mean Value Theorem we see that $\|\mathbb{T}^2(F + H) - \mathbb{T}^2(F)\|_\infty < \kappa \|H\|_\infty$, where $\kappa = \|\Psi\|_\infty/e < 1$. \square

Remark 4. *The bound is tight, for $\lambda > e$ the map \mathbb{T} may have different type of global attractors, see Theorem 5 and Theorem 7 (and Remark 11 in its proof). \square*

3. THE LINEAR PART OF THE OPERATOR \mathbb{T} IS A CONVOLUTION ON THE SET OF DISTRIBUTION FUNCTIONS.

In this section $\Psi = \lambda\Phi$, where Φ is a convolution of nondecreasing functions of an interval $[0, 1]$ with respect to a given probability measure μ . For $\mathcal{D} \ni F(z) : [0, 1] \rightarrow [0, 1]$ and F is not decreasing, F can be extended by zero to the left of 0 and by 1 to the right of 1. We do not assume that $F(0) = 0$, in other words the measure defined by $\int dF$ may have an atom at 0. As it is of no consequence to our result we do not resolve the continuity issue at jump points. The two particular functions $\mathbf{0}, \mathbf{1} \in \mathcal{D}$. Note as curiosity that $\int_0^x d\mu(z) \in \mathcal{D}$.

Remark 5. *We assume that μ has no atom at 1, otherwise by rescaling the interval of arguments we can push the support of μ inside $[0, 1]$. However if μ has no atom at the right-most point of its support, again by rescaling we may assume that the point 1 belongs to the support of the measure μ . In both cases $\int_1^1 d\mu = 0$, and hence we may add the condition $F(1) = 1$, which will be preserved by \mathbb{T} . \square*

Proposition 2 (\mathbb{T} defines a dynamical system on \mathcal{D}). *For $F \in \mathcal{D}$ we have $\mathbb{T}F \in \mathcal{D}$.*

Proof. We know that if Φ is positive $\mathbb{T}F(x) \in [0, 1]$. We have to check that $\mathbb{T}F(x)$ is non decreasing and (if we apply the convention of Remark 5) that $\mathbb{T}F(1) = 1$.

- (1) Φ is a positive linear operator. Clearly if $F \geq 0$, then $\Phi F \geq 0$. Hence for $\lambda > 0$ also $0 \leq \mathbb{T}F \leq 1$.
- (2) If $y - x \geq 0$, then by assumption $F(y) - F(x) \geq 0$ and $F(z - y) - F(z - x) \leq 0$, therefore:

$$\Phi F(y) - \Phi F(x) = \int_y^1 (F(z - y) - F(z - x)) d\mu(z) - \int_x^y F(z - x) d\mu(z) \leq 0.$$

Hence:

$$\mathbb{T}F(y) - \mathbb{T}F(x) = \mathbb{T}F(y) \cdot (1 - \exp(-\lambda(\Phi F(x) - \Phi F(y)))) \geq 0.$$

(3) By Remark 5: $\Phi F(1) = 0$, thus $\mathbb{T}F(1) = \exp(-\lambda \cdot 0) = 1$.

□

Lemma 5 (Φ is self adjoint). *The expected value \mathbb{E}_μ with respect to the measure μ of the convolution $F \star G$ with respect to the Lebesgue measure is equal to:*

$$\int_0^1 F \cdot \Phi G \, dx = \mathbb{E}_\mu[F \star G] = \int_0^1 \Phi F \cdot G \, dx.$$

Proof. We use Fubini Theorem and the change of variables $x = z - w$, all three variables being in $[0, 1]$:

$$\begin{aligned} \int_0^1 F(x) \cdot \Phi G(x) \, dx &= \iint_{z \geq x} F(x) G(z - x) \, d\mu(z) \, dx \\ &= \iint_{w \leq z} F(z - w) G(w) \, dw \, d\mu(z) = \int_0^1 \Phi F(w) \cdot G(w) \, d\mu(z) \, dw. \end{aligned}$$

In fact we could skip the limits as $F(x), G(x)$ are zero for $x < 0$ and the support of μ is in $[0, 1]$. □

Proposition 3.

The linear operator Φ is continuous (and hence differentiable) in the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$, and its operator norm in both cases does not exceed 1.

Proof. We see that $\sup |\Phi(H)| \leq \Phi(\sup |H|) \leq \sup |H|$, where we used that for any function $K(x) \leq k$ with $k > 0$, we have $\Phi(K)(x) \leq k \cdot \mu([x, 1]) \leq k$. Using this inequality with $K(x) = \mathbf{1}(x)$ and Lemma 5 we have:

$$\int_0^1 |\Phi H| \, dx \leq \int_0^1 \mathbf{1} \cdot \Phi |H| \, dx \leq \int_0^1 |H| \cdot \Phi \mathbf{1} \, dx \leq \int_0^1 |H| \, dx.$$

□

Remark 6. *In fact $\|\Phi\|_\infty = 1$ as can be checked by $H = \text{const}$. Also $\|\Phi\|_1 = 1$ if μ has no atom at one (as then, taking H_n constant on $(1 - 1/n, 1]$ leads to the needed estimate). If μ has an atom at one with weight p then $\|\Phi\|_1 \leq 1 - p$. □*

Corollary 5. *Continuity and differentiability of \mathbb{T} follow from Proposition 3 and Lemma 4.*

Theorem 5 (The e -cutoff). *For any $0 < \lambda < e$ and any probability measure μ the map $\mathbb{T}_\lambda(\cdot) = \exp(-\lambda \Phi_\mu(\cdot))$ has a unique fixed point which is a global attractor. On the other hand there exists μ such that for $\lambda > e$ the map \mathbb{T} has no fixed point, which is a global attractor.*

Proof. An example where there is no fixed point global attractor for $\lambda > e$ is presented in Theorem 7 (the case of Dirac measure), see also Remark 11. For $\lambda < e$ the result follows from Theorem 4 with $\Psi = \lambda \Phi$, as $\|\Psi\| = \lambda \|\Phi\| \leq \lambda$. □

Now we present a technical condition which may help to decide on the existence of a globally attracting fixed point.

Theorem 6. *If there exists an N such that*

$$\mathbf{1}^{2N+1}(0) > \frac{1}{e},$$

then \mathbb{T} has a fixed point which is a global attractor in the norm $\|\cdot\|_1$.

Note that we consider an odd iterate of $\mathbf{1}$ or equivalently an even iterate of $\mathbf{0}$.

Proof. From the assumption it follows that $\mathbf{L}(x) \geq \mathbf{L}(0) \geq \mathbf{1}^{2n+1}(0) > 1/e$. The function $x \ln(x)$ is increasing for $x > 1/e$ hence, as $1/e \leq \mathbf{L} \leq \mathbf{U}$ we have $\mathbf{L} \ln \mathbf{L} \leq \mathbf{U} \ln \mathbf{U}$. On the other hand $\ln \mathbf{L} = -\lambda \Phi \mathbf{U}$ and $\ln \mathbf{U} = -\lambda \Phi \mathbf{L}$:

$$0 \leq \int_0^1 (\mathbf{U} \ln \mathbf{U} - \mathbf{L} \ln \mathbf{L}) dx = -\lambda \int_0^1 (\mathbf{U} \cdot \Phi \mathbf{L} - \mathbf{L} \cdot \Phi \mathbf{U}) dx = 0,$$

by Lemma 5. Hence $\|\mathbf{U} - \mathbf{L}\|_1 = 0$ and by Corollary 1, $\mathbf{U} = \mathbf{L}$ is a global attractor in $\|\cdot\|_1$. \square

Remark 7. *The condition given in Theorem 6 is not necessary. When μ is the Lebesgue measure it is shown in Remark 9.* \square

4. EXAMPLES

We consider three examples of the measure μ which defines the convolution Φ : the uniform distribution $d\mu(x) = dx$ on the interval $[0, 1]$, the exponential distribution $d\mu(x) = ae^{-ax}dx$ on the segment \mathbb{R}^+ and the Dirac measure at a point $t \in COInt(0, 1)$, i.e., $\mu(A) = \delta_t(A) = \chi(A)(t)$, where again χ is an indicator function of the set A at the point t .

Theorem 7.

Uniform distribution on $[0, 1]$. *For every $\lambda > 0$, the map \mathbb{T} has a unique fixed point:*

$$\check{F}_A(x) = \frac{Ae^{A\lambda x}}{e^{A\lambda}(A-1) + e^{A\lambda x}},$$

which is a global attractor. Here $1 \leq A = A(\lambda)$ is the unique solution of the equation:

$$e^{A\lambda}(A-1)^2 = 1.$$

Exponential distribution (cf. [GNS03]). *For every $\lambda > 0$ and every a , the map \mathbb{T} has a unique fixed point:*

$$\check{F}_K(x) = \exp(-Ke^{-ax}),$$

which is a global attractor. Here $0 \leq K = K(\lambda)$ (independent on a) is the unique solution of $\check{f}(K) = K$:

$$\check{f}(K) = \lambda \frac{1 - e^{-K}}{K}.$$

Dirac measure δ_t (cf. [KS81]). *For every $0 < \lambda \leq e$ the map \mathbb{T} has a unique fixed point:*

$$\check{F}_M(x) = \begin{cases} 0, & x < 0 \\ M, & 0 < x < t \\ 1, & t < x \end{cases}$$

which is a global attractor. Here M is a unique solution of $\check{f}(M) = M$:

$$\check{f}(M) = e^{-\lambda M}.$$

For every $\lambda > e$ there is no fixed point which is a global attractor.

Proof. In each case we first investigate the properties of the equations on the parameters.

The Lebesgue measure. In this case:

$$\Phi(F)(x) = \int_0^{1-x} F(z) dz.$$

Proposition 4. *The only twice differentiable function satisfying $\mathbb{T}^2(F) = F$, $F(1) = 1$ and F non decreasing is the function \check{F} with $A \geq 1$. It follows that \check{F}_A is a fixed point of \mathbb{T} .*

Proof. We want to solve the equation $\mathbb{T}^2(F) = F$, i.e.

$$e^{-\lambda \int_0^{1-x} e^{-\lambda \int_0^{1-z} F(s) ds} dz} = F(x).$$

After applying \ln to both sides of the above equation and differentiating them with respect to x we get

$$(4) \quad \lambda e^{-\lambda \int_0^x F(s) ds} = \frac{F'(x)}{F(x)}.$$

We repeat the same procedure once again, and we obtain the second-order differential equation:

$$-\lambda F(x) = \frac{F''(x)F(x) - (F'(x))^2}{F'(x)F(x)},$$

in other words

$$F''F - (F')^2 + \lambda F'F^2 = 0.$$

This equation does not contain the independent variable, so the standard substitution $z = F'(F)$ allows us to lower the degree of the equation to 1. Easily we get the solution $F(x) = \frac{CAe^{A\lambda x}}{1 + Ce^{A\lambda x}}$. From $F(1) = 1$ there follows $C = \frac{e^{-A\lambda}}{A-1}$, and

$$F(x) = \frac{Ae^{A\lambda x}}{e^{A\lambda}(A-1) + e^{A\lambda x}},$$

and from $F(x) \geq 0$, $F'(x) \geq 0$ on $[0, 1]$ there follows $A \geq 1$. We substitute this function into the equation (4), and obtain:

$$\frac{(1 + (A-1)e^{A\lambda})\lambda}{(A-1)e^{A\lambda} + e^{A\lambda x}} = \frac{(A-1)Ae^{A\lambda}\lambda}{(A-1)e^{A\lambda} + e^{A\lambda x}},$$

so $(A-1)^2 e^{A\lambda} = 1$. By assumption $F = \check{F}_A$ is a periodic point of period two. Because \mathbb{T} preserves both additional conditions $\mathbb{T}\check{F}_A$ fulfils the assumptions of the proposition and hence $\mathbb{T}\check{F}_A = \check{F}_A$. \square

The monotone maps \mathbf{L} and \mathbf{U} are the images of themselves under the map \mathbb{T} which is a composition of a smooth exponential map with the convolution Φ with the smooth kernel 1. That means that \mathbf{L} and \mathbf{U} are at least twice differentiable and hence satisfy the assumption of Proposition 4, so must be both equal to the map \check{F} .

Remark 8. *It is possible to construct and solve a differential equation for the fixed point of \mathbb{T} . However as the convolution Φ exchanges the argument x into $1 - x$ one gets:*

$$-\lambda \frac{d}{dx} \Phi F(x) = \lambda F(1 - x) = \frac{F'(x)}{F(x)},$$

which is not a differential equation. One has to use special symmetries to get rid of $F(1 - x)$. Indeed, integrating and using $F(1) = 1$ one obtains:

$$F(x) = \lambda \int_0^x F(z)F(1 - z)dz + 1 - k,$$

which produces the needed formula for $F(1 - x)$. \square

Remark 9. *After simplifications we have $\check{F}_A(0) = A - 1$, where A runs from one to two as λ runs from infinity to zero. For $\lambda > e/(e + 1)$ we have $A - 1 < 1/e$ which shows that the condition in Theorem 6 is not necessary.* \square

The exponential measure. Here we work with a slightly different setting, namely the interval of the arguments in the definition of \mathcal{D} becomes the segment $(0, +\infty)$. There is a simplification in the convolution:

$$\Phi(F)(x) = \int_x^\infty F(z - x)ae^{-az} dz = e^{-ax} \int_0^\infty F(w)ae^{-aw} dw = e^{-ax} \mathbb{E}_a[F],$$

which means that dependance on x is outside the integral, so that we can write:

$$\mathbb{T}(F)(x) = \exp(-\lambda e^{-ax} \int_0^\infty F(z)ae^{-az} dz) = \exp(-\lambda e^{-ax} \mathbb{E}_a[F]).$$

If we set $G(x) = F(x/a)$ then we get a conjugated evolution which is independent on a :

$$\check{\mathbb{T}}(G)(x) = \exp(-\lambda e^{-x} \mathbb{E}[G]), \quad \text{where} \quad \mathbb{E}[G] = \mathbb{E}_1[G] = \int_0^\infty G(z)e^{-z} dz.$$

In other words after one iteration the collection of distributions consists of one parameter family of double exponential functions $\check{\mathcal{D}} = \{\exp(-Ke^{-x}), K > 0\}$, and the dynamic in the space of functions is reduced to the dynamics of the parameter K . For $G \in \check{\mathcal{D}}$ with parameter K we have $\check{\mathbb{T}}G \in \check{\mathcal{D}}$ with parameter $\lambda \mathbb{E}G$, or $K \mapsto \check{f}(K) = \lambda \mathbb{E}G$:

$$\check{f}(K) = \lambda \int_0^\infty G(z)e^{-z} dz = \lambda \int_0^1 \exp(-Kt) dt = \lambda \frac{1 - e^{-K}}{K}.$$

In particular the equation $\check{f}(\check{f}(K)) = K$ has a unique positive solution (which is also a solution of $\check{f}(K) = K$), which means that \mathbb{T} has no periodic points of proper period two, or that $\mathbf{L} = \mathbf{U}$.

Remark 10. *Also here for $\lambda > e/(e - 1)$ we have $K > 1$ and hence $\check{F}(0) = e^{-K} < 1/e$.*

4.1. The Dirac measure.

Lemma 6. *For each λ the real function $\tilde{f}(x) = e^{-\lambda x}$ has a unique fixed point $M = M(\lambda)$. For $\lambda \leq e$, the point M is an attractor, for $\lambda > e$ it is a repeller, but then \tilde{f} has an attracting periodic orbit which attracts every $x \neq M$.*

Proof. The function $f_\lambda(x)$ is decreasing, so the first statement is obvious. Using Implicit Function Theorem one checks that $dM/d\lambda < 0$. We have $\tilde{f}'(M) < 0$, and $(d/d\lambda)(\tilde{f}'(M)) < 0$, so the derivative of \tilde{f} at M is also a decreasing function of λ . The equation $\tilde{f}'(M) = -1$, that is $-\lambda e^{-\lambda M} = -\lambda M = -1$ has a unique solution with $\lambda = e$ and $M = 1/e$. In order to check that for $\lambda \leq e$ this gives the global attractor consider the second iterate, and see that for $\lambda \leq e$, $(\tilde{f}(\tilde{f}(x)) - x)(M - x) > 0$ for $x \neq M$, and that for $\lambda > e$ there is an orbit of period 2 attracting all $x \neq M$. The details of this exercise are omitted. \square

The explicit form of $\mathbb{T}F$ in the case of $\mu = \delta_b$ is given by:

$$\mathbb{T}F(x) = \exp(-\lambda \int_x^1 F(z-x)\delta_b(z)) = \begin{cases} e^{-\lambda F(b-x)} & x \in [0, b] \\ 1 & x \in (b, 1] \end{cases}.$$

We observe that if F is constant on the interval $[0, b]$ so is $\mathbb{T}(F)$. One can easily check that

$$\mathbf{1}^n(x) = \begin{cases} \tilde{f}^{n+1}(0) & x \in [0, b] \\ 1 & x \in (b, 1] \end{cases},$$

and the theorem follows from Lemma 6.

Remark 11. For $\lambda > e$ the points $L < M < U$ are the two periodic points of period two which attracts (almost) every trajectory of \tilde{f} . For $F \leq M$ we have $\mathbb{T}^{2n}F \rightarrow L$ and $\mathbb{T}^{2n+1}F \rightarrow U$ (for $F \geq U$ we have an analogous statement) point-wise at every point $x \leq b$. If $F(x) = M$ for some $x \leq b$ then $\mathbb{T}^{2n}F(x) = M$ and $\mathbb{T}^{2n+1}F(b-x) = M$. Let J be a minimal interval which contains all such points x and $a-x$, then outside J the iterates accumulate on L and U while inside J the accumulation points swap between L , M and U depending on the values of F at points $a-x$ (but preserving the monotony of $\mathbb{T}^n F$). This shows that there is no simple attracting point for all $F \in \mathcal{D}$ and shows that the bound in Theorem 5 is tight. The details are skipped. \square

Remark 12. The dynamics get much more complicated already for $\mu = p\delta_a + (1-p)\delta_b$. There is a partition of $[0, 1]$ into $N = 2/(b-a)$ intervals of the points $m(b-a)$ and $b-m(b-a)$, $m = 0, 1, \dots$ such that a subset of \mathcal{D} of maps constant on these intervals is invariant under \mathbb{T} (independently on λ). Studying this invariant subset is sufficient as that is where all the iterates of $\mathbf{1}$ live. This reduces the dynamics in the functional space into the dynamics in \mathbb{R}^N . Similarly for $\mu = \sum p_i \delta_i$. In the simplest case $a_i = (i-1)/n$ the invariant partition consists of n intervals, and the resulting dynamical system is a map from $[0, 1]^n$ into itself. \square

This concludes the description of the examples and the proof of Theorem 7. \square

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